Analytic model for a frictional shallow-water undular bore

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Abstract

We use the integrable Kaup-Boussinesq shallow water system, modified by a small viscous term, to model the formation of an undular bore with a steady profile. The description is made in terms of the corresponding integrable Whitham system, also appropriately modified by friction. This is derived in Riemann variables using a modified finite-gap integration technique for the AKNS scheme. The Whitham system is then reduced to a simple first-order differential equation which is integrated numerically to obtain an asymptotic profile of the undular bore, with the local oscillatory structure described by the periodic solution of the unperturbed Kaup-Boussinesq system. This solution of the Whitham equations is shown to be consistent with certain jump conditions following directly from conservation laws for the original system. A comparison is made with the recently studied dissipationless case for the same system, where the undular bore is unsteady.

Undular bores are nonlinear wave-like structures, which are generated in the breaking profiles of large-scale nonlinear waves propagating in dispersve media. A general theory, based on the Whitham modulation equations, has been previously developed for dissipationless, unsteady, undular bores on the basis of completely integrable models such as the Korteweg-de Vries equation, nonlinear Schrödinger equation etc. The introduction of physically important small dissipation in the system dramatically changes its properties, allowing in some cases for the presence of steady solutions. The most explored model

describing the effects of friction on an undular bore is based on the uni-directional Korteweg-de Vries equation, modified by a small friction term, which can take various forms. Appropriate perturbation techniques have been used to obtain asymptotic solutions. However, unlike the case for conservative undular bores, no general approach seems to be available. Here, using an integrable version of the bi-directional Boussinesq equations, but modified by a small Burgers-like dissipation term, we develop a modulation theory of frictional shallow water undular bores, which can also be extended to other non-conservatively perturbed integrable systems.

1 Introduction

It is well known that solution to an initial value problem for the inviscid dispersionless shallow water equations may develop a wave-breaking singularity after a finite time, when the first spatial derivatives become infinite. After the wave-breaking point, a formal solution becomes multi-valued and loses its physical meaning. The divergence of the spatial derivatives at the wave-breaking point suggests that higher-order terms must be taken into account. These terms can be either dispersive or dissipative in nature, or, as here, a combination of both. The form of the solution after the breaking time then strongly depends on the comparative values of the dispersion and dissipative terms. If dissipation can be neglected in favour of dispersion, the solution in a certain neighbourhood of the breaking point assumes the form of an expanding nonlinear oscillatory structure with a solitary wave train generated in the vicinity of its leading edge. This structure provides a dispersive resolution of a breaking singularity, and is an unsteady undular bore (or a dispersive shock in a different terminology). Unsteady undular bores have been studied extensively in the last thirty years on the basis of exactly integrable nonlinear wave equations. The original formulation of the problem was given by Gurevich and Pitaevskii (1973) who proposed to describe the expanding collisionless shocks (a plasma analog of undular bores) with the aid of the Whitham-averaged equations for the integrable Korteweg-de Vries (KdV) equation. The Gurevich-Pitaevskii theory has been extended to other important integrable systems such as the nonlinear Schrödinger and Kaup-Boussinesq equations (see Kamchatnov (2000), for instance, for the detailed account on the Gurevich-Pitaevskii theory).

The introduction of small dissipation can, in some cases, balance the dispersive effects so that the undular bore eventually acquires a steady profile, but remains oscillatory in space. The analytical study of steady (frictional) undular bores was initiated in the classical work of Benjamin and Lighthill (1954) on shallow water waves. Another important work on the same subject, but in the context of collisionless plasma shocks with small dissipation, is Sagdeev (1964). In both works the authors use a mechanical analogy with a weakly damped nonlinear oscillator to explain the main observable features of undular bores: the formation of the lead solitary wave and degeneration into linear sinusoidal waves at the rear. It was also suggested that the undular bore transition conditions must be consistent with the conservation of mass and momentum across the transition zone, while the violation of hydrodynamic energy conservation is remedied by taking into account the generated waves.

A simple model with an analytic description of shallow water frictional undular bore is provided by travelling wave solutions of the KdV-Burgers equation (see for instance

Whitham(1974))

$$u_t + uu_x + u_{xxx} = \nu u_{xx} \tag{1}$$

with a small dissipation coefficient $0 < \nu \ll 1$. A detailed study of such solutions was made by Johnson (1970), who applied a direct perturbation procedure (Kuzmak 1959) to the periodic solution of the unperturbed ($\nu = 0$) equation (the KdV cnoidal wave), and performed matching of the leading order approximation with the solitary wave to obtain a closed description. Johnson's (1970) solution has been used by Smyth (1988) for the description of the effect of small dissipation on resonant flow over topography.

The description of the undular bore on the basis of the steady travelling wave solutions of the KdV-Burgers equation has two inherent restrictions: (i) as it is based on a unidirectional equation, it does not reveal the transition (jump) conditions across the undular bore and can link any two given constant states $u = u_2$ and $u = u_1$, $u_2 > u_1$; (ii) it describes only the established (steady) regime, and says nothing about the undular bore formation. The further development of the Whitham modulation theory (Whitham (1965, 1974)) in 1970-80s due to Gurevich and Pitaevskii (1973, 1987), Lax, Levermore and Venakides (see the review (1994) and references therein), Flaschka, Forest and McLaughlin (1982), Dubrovin and Novikov (see the review (1989) and references therein) and many other authors, has made it clear that the consistent description of undular bores (both conservative and frictional) should be made in the framework of the hydrodynamic-type Whitham equations describing the evolution of nonlinear modulated waves. Although the Whitham equations, based on averaging over the periodic wave family, correspond to the leading order of a direct perturbation procedure, which formally diverges when the wave period tends to infinity (the solitary wave limit), their solutions reveal only a weak singularity at the leading edge of the undular bore (see Gurevich and Pitaevskii (1973, 1987)) and yield the correct value for the lead solitary wave amplitude (while its position, of course, is not determined accurately). Also, the Whitham equations have been shown to inherit an integrable (or perturbed integrable) structure from the original system and allow in some cases the effective construction of exact *global* solutions using powerful methods developed in the theory of finite-gap integration, and in the theory of integrable Hamiltonian systems of hydrodynamic type (Tsarev (1985, 1990), Dubrovin and Novikov (1989)).

The modulation theory of the "integrable" shallow water undular bore has been constructed by El, Grimshaw and Pavlov (2001) using the extension to the case of the bi-directional Kaup-Boussinesq system of the original formulation of Gurevich and Pitaevskii (1974) for the decay of an initial discontinuity in the KdV equation. A more general case of the formation of an undular bore in the vicinity of a "cubic" breaking point has been studied in (El, Grimshaw and Kamchatnov, 2005). An asymptotic theory of the formation of soliton trains from a "big" enough initial pulse for the Kaup-Boussinesq system was developed by Kamchatnov, Kraenkel and Umarov, 2003.

The modulation equations for the KdV-Burgers equation were derived by Gurevich and Pitaevskii (1987) and Avilov, Krichever and Novikov (1987). Other forms of the dissipative term have been considered by Gurevich and Pitaevskii (1991) (boundary layer-type), and Myint and Grimshaw (1995) (boundary layer dissipation and Rayleigh friction). In their work on the modulation theory of the KdV-Burgers equation, Gurevich and Pitaevskii (1987) deduced an exact steady solution of the Whitham system corresponding to the steady undular bore. It is worth noting that their solution exactly coincides with the leading-order perturbation solution to (1) obtained much earlier by Johnson (1970). Avilov, Krichever and

Novikov (1987) have shown numerically that this solution is indeed the large-time asymptotic modulation solution of (1) with the initial conditions in the form of a smooth step.

A general procedure for obtaining perturbed modulation system for the KdV equation, based on the finite-gap integration machinery, was formulated by Forest and McLaughlin (1984). A more effective method for the case of periodic modulated waves, applicable not only to the perturbed KdV equation, but also to the whole perturbed AKNS hierarchy has recently been designed by Kamchatnov (2004).

In this paper, we apply Kamchatnov's procedure to the Kaup-Boussinesq system modified by a small Burgers-like dissipative term. Being bi-directional, this system allows a more realistic modelling of frictional undular bores than the KdV-Burgers equation (1). We distinguish several characteristic stages of the undular bore evolution and construct exact solutions of the Whitham equations describing the initial (unsteady) and final (steady) stages of evolution of the KBB undular bore. The methods used in this paper also allow for the analytic description of undular bores generated in presence of an external force. In particular, an important class of problems of this kind occurs in the description of resonant flow over topography (see Grimshaw & Smyth (1986) and Smyth (1987, 1988)).

2 Formation of a frictional undular bore: general description

We consider formation of a frictional undular bore in the Kaup-Boussinesq system modified by a small viscous term. In dimensionless variables this system has the form:

$$\begin{aligned} &h_t + (hu)_x + \tfrac{1}{4}u_{xxx} = 0, \\ &u_t + uu_x + h_x = \nu u_{xx}, \end{aligned} \tag{2}$$

where $h(x,t)$ denotes the height of the water surface above a horizontal bottom, $u(x,t)$ is related to the horizontal velocity field averaged over depth (see (Kaup (1976) for the detailed derivation of the inviscid system) and $0 < \nu \ll 1$ is a small viscosity coefficient. We shall call Eq. (2) the Kaup-Boussinesq-Burgers (KBB) system.

Note that the frictional term appears only in the second equation, which represents the momentum balance, and is absent in the first equation which represents the mass balance. Also, the derivation of this system requires that the frictional term is a small term, of the same order as the small dispersion term. In the sequel, however, we will be treating the frictional term as a small perturbation to an inviscid system.

Compared to the KdV-Burgers equation (1), the KBB system (2) has the essential advantage of modelling bi-directional wave propagation, so the undular bore description would necessarily include transition conditions, which should be consistent with the jump conditions following from the conservation laws of the system (2). On the other hand, the KBB system (2) is a perturbed integrable system, which retains the advantage of amenability to an effective analytic study. A drawback of the KBB system as a model system is the presence of a high-wavenumber instability of the constant solutions. That is, the linearized equations allow for growing waves at large wavenumbers (see El, Grimshaw, & Pavlov 2001).

Also, there is the disadvantage for the description of undular bores that there is no "physical" momentum conservation law, which leads to formally "nonphysical" transition

conditions. We will show, however, that the transition conditions following from the solutions of Eq. (2) are asymptotically consistent with the classical jump conditions for shallow water bores within the range of applicability of the KBB system.

We consider initial data at $t = t_0$: $h_0(x) = h(x, t_0)$, $u_0(x) = u(x, t_0)$ for the system (2) in the form of a smooth transition between two constant states:

$$\begin{aligned} h = h_1, \quad u = u_1 \qquad &\text{as} \;\; x \to +\infty, \\ h = h_2, \quad u = u_2 \qquad &\text{as} \;\; x \to -\infty, \end{aligned} \tag{3}$$

so that $h_2 > h_1$ and the characteristic width of the transition region $l \gg 1$. There are two typical spatio-temporal scales associated with the initial-value problem (2), (3): characteristic "nonlinear-dispersive" scale $\Delta t_{nlin} \sim \Delta x_{nlin} \sim l$ and the "dissipative" scale $\Delta t_d \sim \Delta x_d \sim \nu^{-1}$. Let us suppose that the initial conditions are chosen such that $t_d \gg t_{nlin}$, i.e. we have

$$l \gg 1, \qquad \nu \ll 1, \quad \text{and} \qquad \nu l \ll 1\,. \tag{4}$$

Then, following Avilov, Krichever and Novikov (1987), we distinguish several stages in the process of the formation of a frictional undular bore, and discuss some limitations of the applicability of this scenario.

Stage 1. $t_0 < t < t_{br} \ll \nu^{-1}$, where the breaking time t_{br} will be defined below; $t_{br} - t_0 \sim l$. Due to Eq. (4) the initial data satisfy the following inequalities:

$$\frac{1}{4}|u_0'''| \ll |(h_0 u_0)'|, \qquad \nu|u_0''| \ll |u_0 u_0'|, \tag{5}$$

Thus this stage of the evolution can be described by the ideal shallow water system

$$h_t + (hu)_x = 0, \qquad u_t + uu_x + h_x = 0, \tag{6}$$

which can be represented in the diagonal form

$$\frac{\partial \lambda_+}{\partial t} + \frac{1}{2}(3\lambda_+ + \lambda_-)\frac{\partial \lambda_+}{\partial x} = 0, \quad \frac{\partial \lambda_-}{\partial t} + \frac{1}{2}(\lambda_+ + 3\lambda_-)\frac{\partial \lambda_-}{\partial x} = 0. \tag{7}$$

Here

$$\lambda_\pm = \frac{u}{2} \pm \sqrt{h} \tag{8}$$

are the Riemann invariants of Eqs. (6).

Initial data are given by two functions $\lambda_+(x, t_0)$ and $\lambda_-(x, t_0)$ determined by the initial distributions $h_0(x)$ and $u_0(x)$. The system (7) has two families of characteristics in the (x, t) plane along which one of two Riemann invariants (either λ_+ or λ_-) is constant. The wave-breaking point corresponds to the moment when characteristics of one of the families begin to intersect, so that the corresponding Riemann invariant becomes a three-valued function in the physical plane. Let such an intersection occur for the characteristics transferring the values of λ_+. Then at the wave-breaking point the profile of λ_+ as a function of x has a vertical tangent line and, hence, in vicinity of this point it varies very fast, whereas the second Riemann invariant varies with x much slower and may be considered here as a constant parameter:

$$\lambda_- = \lambda_0 = \text{const.} \tag{9}$$

Thus, in the vicinity of the breaking point at $t = t_{br}$ we are dealing with a simple wave. Without loss of generality we choose $t_{br} = 0$. The second equation in (7) is identically satisfied by Eq. (9). The first equation in (7) then has the well-known solution

$$x - \frac{1}{2}(3\lambda_+ + \lambda_0)t = f(\lambda_+), \tag{10}$$

where $f(\lambda_+)$ is an inverse function to an initial profile $\lambda_+(x,0)$. Due to our normalization, the function $x = f(\lambda_+)$ must have an inflexion point with a vertical tangent line at $t = 0$. In the vicinity of this point $f(\lambda_+)$ can be approximated by a cubic function,

$$x - \frac{1}{2}(3\lambda_+ + \lambda_0)t = -C(\lambda_+ - \lambda_+^0)^3, \tag{11}$$

where C and λ_+^0 are constants. Without loss of generality Eq. (11) can be cast into the form (see El, Grimshaw and Kamchatnov (2005) for details)

$$x - \frac{1}{2}(3\lambda_+ + \lambda_0)t = -\lambda_+^3. \tag{12}$$

It corresponds to the wave breaking picture shown in Fig. 1.

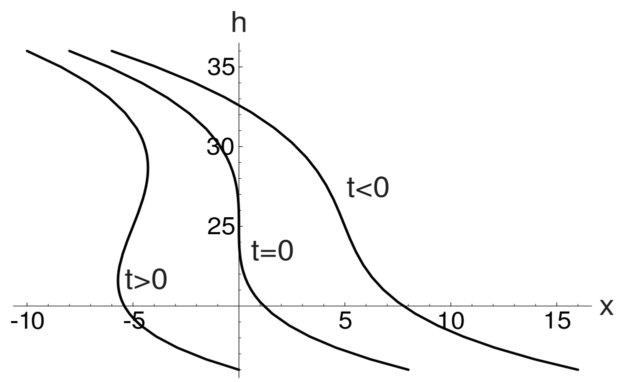

Figure 1: Wave breaking of the water elevation in the dispersionless limit; λ_0 is taken equal to -10.

Stage 2. $t_{br} < t \ll \nu^{-1}$. At this stage, dispersion should be taken into account in the vicinity of the breaking point which implies consideration of the integrable KB system:

$$h_t + (hu)_x + \frac{1}{4}u_{xxx} = 0, \qquad u_t + uu_x + h_x = 0. \tag{13}$$

with the initial data following from (12), (8), (9):

$$t = 0: \qquad \frac{u}{2} + \sqrt{h} = -x^{1/3}; \quad \frac{u}{2} - \sqrt{h} = \lambda_0\,. \tag{14}$$

The combined action of nonlinearity and dispersion leads to the generation of an expanding nonlinear oscillatory structure occupying the finite space interval $(x^-; x^+)$ (see Fig. 2). This structure is an unsteady, "conservative" undular bore. Outside the interval $(x^-; x^+)$ the flow is smooth and is described by the solution (9), (12). The solution of the problem now

consists of two parts. Following Gurevich and Pitaevskii (1973), we suppose that the region of oscillations can be approximated by a modulated periodic solution of the KB system. Its global evolution is then determined by the Whitham equations and the problem reduces to finding the solution of the Whitham equations that matches the solution (12) at the end points of the oscillatory region. One may say that this oscillatory region (the expanding undular bore) "replaces" a non-physical multi-valued region of the solution (12). One should emphasize, however, that the boundaries of the undular bore $x^{\pm}$ *do not coincide* with the boundaries of the formal multi-valued solution.

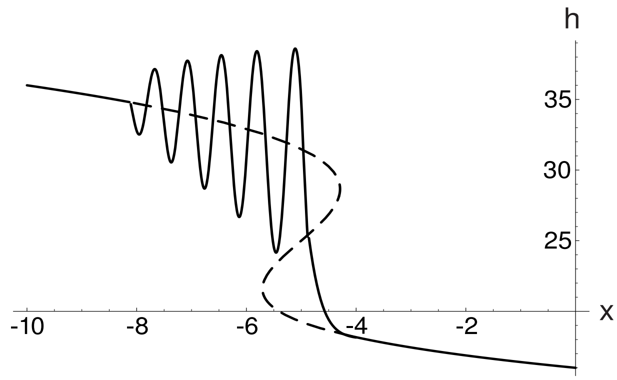

Figure 2: Initial stage of the undular bore development. The plot corresponds to the time $t = 1$ and $\lambda_1 = -10$. Dashed line shows the formal solution in the dispersionless limit.

The corresponding modulated solution of the KB system has been constructed by El, Grimshaw and Kamchatnov (2005). Here we briefly outline the resulting formulas. The derivation of the complete modulation system with the account of the dissipative corrections will be presented in Section 4.

The local wave form of the undular bore is given by the periodic travelling wave solution of the KB system (13), and is given by the expressions

$$u(x,t) = s_1 - 2\mu(\theta), \quad h(x,t) = \tfrac{1}{4}s_1^2 - s_2 - 2\mu^2(\theta) + s_1\mu(\theta), \quad \theta = x - \tfrac{1}{2}s_1 t, \tag{15}$$

where

$$\mu(\theta) = \frac{\lambda_2(\lambda_3-\lambda_1) - \lambda_1(\lambda_3-\lambda_2)\mathrm{sn}^2\left(\sqrt{(\lambda_4-\lambda_2)(\lambda_3-\lambda_1)}\,\theta, m\right)}{\lambda_3-\lambda_1-(\lambda_3-\lambda_2)\mathrm{sn}^2\left(\sqrt{(\lambda_4-\lambda_2)(\lambda_3-\lambda_1)}\,\theta, m\right)}. \tag{16}$$

$$\lambda_4 \geq \lambda_3 \geq \lambda_2 \geq \lambda_1.$$

Here $\mathrm{sn}(\theta, m)$ is the Jacobi elliptic function with the modulus

$$m = \frac{(\lambda_3-\lambda_2)(\lambda_4-\lambda_1)}{(\lambda_4-\lambda_2)(\lambda_3-\lambda_1)}. \tag{17}$$

The connection of the constants s_1, s_2 in Eq. (15) with the parameters λ_j in Eq. (16) is given by

$$s_1 = \sum_{j=1}^{4}\lambda_j\,, \qquad s_2 = \sum_{i<j}\lambda_i\lambda_j\,. \tag{18}$$

The soliton limit ($m = 1$) is obtained either for $\lambda_1 = \lambda_2$ or for $\lambda_3 = \lambda_4$. For $\lambda_3 = \lambda_4$, which corresponds to right-propagating solitons, Eq. (16) yields

$$\mu(\theta) = \lambda_1 + \frac{(\lambda_2 - \lambda_1)(\lambda_4 - \lambda_1)}{\lambda_2 - \lambda_1 + (\lambda_4 - \lambda_2)/\cosh^2[\sqrt{(\lambda_4 - \lambda_2)(\lambda_4 - \lambda_1)}\,\theta]}, \tag{19}$$

Modulations $\lambda_i(x, t)$ in the travelling wave solution are described by the Whitham equations, which have been derived for the KB system by El, Grimshaw, and Pavlov (2001) in the Riemann form (see also El, Grimshaw and Kamchatnov (2005)).

$$\frac{\partial \lambda_i}{\partial t} + v_i \frac{\partial \lambda_i}{\partial x} = 0, \quad i = 1, 2, 3, 4. \tag{20}$$

Here the characteristic velocities v_i are expressed in terms of λ_j as

$$v_i = \left(1 - \frac{L}{\partial_i L}\partial_i\right) V, \quad \partial_i \equiv \frac{\partial}{\partial \lambda_i}, \quad i = 1, 2, 3, 4, \tag{21}$$

where the phase velocity V and the wavelength L are given correspondingly by

$$V = \frac{1}{2}\sum_{i=1}^{4} \lambda_i\,, \qquad L = \int_{\lambda_2}^{\lambda_3} \frac{d\mu}{\sqrt{P(\mu)}} = \frac{2\mathrm{K}(m)}{\sqrt{(\lambda_4 - \lambda_2)(\lambda_3 - \lambda_1)}}, \tag{22}$$

$\mathrm{K}(m)$ being the complete elliptic integral of the first kind.

The solution of the Whitham system (20) matching the dispersionless solution (9), (12) is obtained using the generalized hodograph transform (Tsarev 1985, 1990) and has the form

$$\begin{aligned} x - v_i t &= -\tfrac{16}{35} w_i^{(3)} + \tfrac{8}{35}\lambda_0 w_i^{(2)} + \tfrac{2}{35}{\lambda_0}^2 w_i^{(1)} + \tfrac{1}{35}{\lambda_0}^3, \quad i = 2, 3, 4; \\ \lambda_1 &= \lambda_0 = \mathrm{const}, \end{aligned} \tag{23}$$

where

$$w_i^{(k)} = \left(1 - \frac{L}{\partial_i L}\partial_i\right) W^{(k)}, \quad i = 1, 2, 3, 4. \tag{24}$$

Here the functions $W^{(k)}(\lambda_1, \ldots, \lambda_4)$ are obtained as coefficients of the series expansion

$$W = \frac{\lambda^2}{\sqrt{\prod_{j=1}^{4}(\lambda - \lambda_j)}} = \sum \frac{W^{(k)}}{\lambda^k} = 1 + \tfrac{1}{2}s_1 \cdot \frac{1}{\lambda} + \left(\tfrac{3}{8}s_1^2 - \tfrac{1}{2}s_2\right) \cdot \frac{1}{\lambda^2} + \left(\tfrac{5}{16}s_1^3 - \tfrac{3}{4}s_1 s_2 + \tfrac{1}{2}s_3\right) \cdot \frac{1}{\lambda^3} + \ldots. \tag{25}$$

One can see that $w_i^{(1)} = v_i$ coincide with the characteristic velocities (21).

Formulas (23)–(25) define $\lambda_2, \lambda_3, \lambda_4$ implicitly as functions of x and t and, together with the travelling wave solution (15)–(16), determine the evolution of the undular bore at stage 2. Dependence of the Riemann invariants on x at some fixed moment of time is shown in Fig. 3.

The dynamics of the edges $x^{\pm}(t)$ of the undular bore at this stage 2 is given by the formulas

$$x^{+}(t) = \frac{1}{2}\lambda_0 t + \frac{1}{6}\sqrt{\frac{5}{3}}\, t^{3/2}, \tag{26}$$

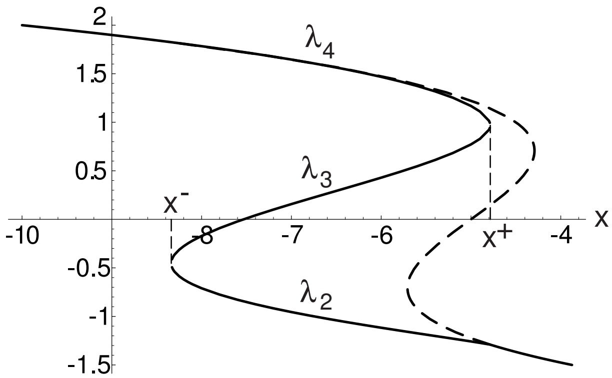

Figure 3: Dependence of Riemann invariants $\lambda_2, \lambda_3, \lambda_4$ on x at fixed time $t = 1$ and with $\lambda_1 = -10$. The dashed line shows the corresponding dependence of λ_+ for the formal multi-valued solution of the KB equations in the dispersionless limit.

$$x^-(t) \cong \frac{1}{2}\lambda_0 t - \frac{3\sqrt{3}}{2}t^{3/2} + \frac{75}{14}\frac{t^2}{\lambda_0}, \qquad \sqrt{3t} \ll |\lambda_0|. \tag{27}$$

We note that this solution is generically realized only at the initial stage of the undular bore development, where the cubic approximation (14) of the initial function is valid. After that, one should use the solution of the Whitham equations corresponding to the actual initial data. Such a solution can also be constructed in a closed form using the generalized hodograph method (see Gurevich, Krylov, El (1992) for the KdV case and El and Krylov (1995) for the defocusing NLS case).

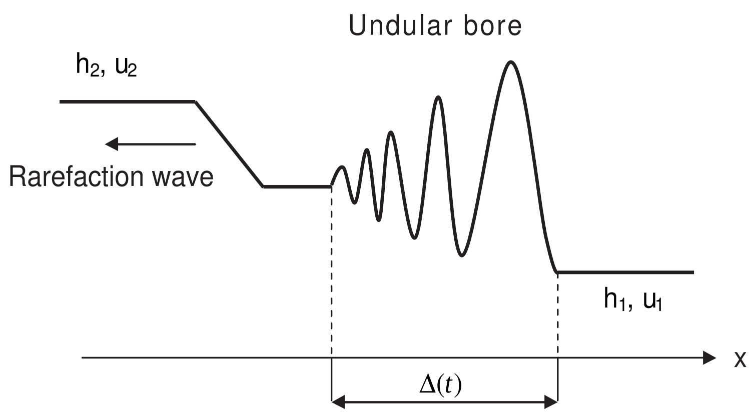

Figure 4: Formation of an unsteady undular bore and rarefaction wave as an intermediate asymptotic for $l < \Delta(t) < \nu^{-1}$.

Stage 2a. If the dissipation coefficient ν is small enough such that for some time interval the following inequality holds: $l \ll \Delta(t) \ll \nu^{-1}$, where $\Delta(t) = x^+(t) - x^-(t)$ is the undular

bore width, then an intermediate similarity asymptotic for the undular bore is realized where all the modulation parameters λ_j in (20) are functions of $\tau = x/t$ only. These solutions have been studied in detail in (El, Grimshaw and Pavlov 2001) where the problem of the decay of an initial discontinuity for the KB system has been considered. Generally, owing to the two-wave nature of the KB system such a solution would involve a rear rarefaction wave along with the leading undular bore (Fig. 4). The similarity solution of the Whitham equations (20) in the undular bore region has the form:

$$v_3 = \frac{x}{t}; \quad \lambda_1 = \lambda_-^{(1)} \quad \lambda_4 = \lambda_+^{(2)}, \tag{28}$$

while the rarefaction wave is described by the similarity solution of the ideal shallow water equations (7):

$$\frac{1}{2}(\lambda_+ + 3\lambda_-) = \frac{x}{t}, \qquad \lambda_+ = \lambda_+^{(2)} \tag{29}$$

in the interval where

$$\lambda_-^{(2)} \leqslant \lambda_- \leqslant \lambda_-^{(1)}. \tag{30}$$

so that it matches a plateau region $\{\lambda_+ = \lambda_+^{(2)};\ \lambda_- = \lambda_-^{(1)}\}$ at its leading edge and the boundary values $\{\lambda_+ = \lambda_+^{(2)};\ \lambda_- = \lambda_-^{(2)}\}$ at the trailing edge (see the diagram in Fig. 5.). Here

$$\lambda_\pm^{(1)} = \frac{u_1}{2} \pm \sqrt{h_1}, \quad \lambda_\pm^{(2)} = \frac{u_2}{2} \pm \sqrt{h_2}, \tag{31}$$

It follows from the solution (28)–(31) and relation (8) that two given constant states (h_1, u_1) and (h_2, u_2), $h_2 > h_1$ could be connected with the aid of a single *dissipationless* undular bore (i.e. with no rarefaction wave generated), provided the following condition is satisfied

$$\frac{u_2}{2} - \sqrt{h_2} = \frac{u_1}{2} - \sqrt{h_1}. \tag{32}$$

One can notice that this condition (transition relation) coincides with the relationship between flow parameters at any two points in the formal simple wave solution for the ideal shallow water equations (6). At the same time, as we stressed before, the solution of the Whitham equations does not coincide with the three-valued simple wave solution of the shallow water equations.

We emphasize that the similarity stage of the undular bore evolution may not be realized at all if the dissipation coefficient is not small enough (see discussion in Avilov, Krichever and Novikov (1987) for the KdV-Burgers case).

Stage 3. $t \sim \nu^{-1}$. At this stage, the dissipation effects are accumulated to the degree that they begin to compete with the combined action of nonlinearity and dispersion. The dynamics of the undular bore is governed now by the full KBB system (2). The local wave form in the undular bore is still described by the periodic solution (15), (16) but the Whitham equations now become inhomogeneous

$$\frac{\partial \lambda_i}{\partial t} + v_i \frac{\partial \lambda_i}{\partial x} = \rho_i, \quad i = 1, 2, 3, 4. \tag{33}$$

Explicit expressions for the functions $\rho_i(\lambda_1, \ldots, \lambda_4)$ will be derived in Section 4 of this paper. We note that the undular bore at this stage is still unsteady.

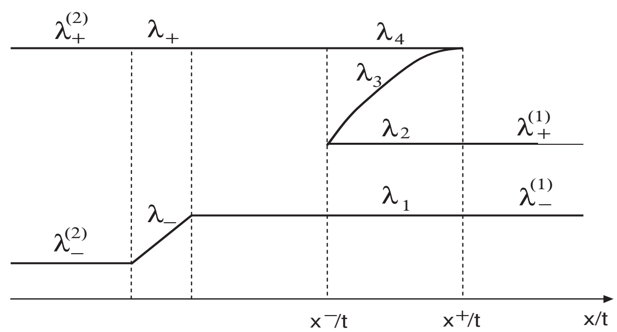

Figure 5: Riemann invariant behaviour in the similarity asymptotic of the modulation solution.

Stage 4. $t \gg \nu^{-1}$. At this stage, the undular bore is reaching its steady regime so that it propagates as a whole with a single velocity, say c. The corresponding solution $\lambda_j = \lambda_j(x-ct)$ of the perturbed Whitham equations (33) will be constructed in Section 5. Nevertheless, some important relationships for the steady frictional undular bore can be obtained from very general reasoning.

Indeed, the original procedure for the derivation of the Whitham equations (Whitham 1965) implies averaging the conservation laws of the original system over the periodic solutions. In the case of non-conservatively perturbed systems (which is the case for the KBB system) the averaging is performed over the periodic family of the *unperturbed* KB system (13).

In our case, we have only two conservation laws for the KBB system (2) at our disposal:

$$h_t + (hu + \tfrac{1}{4}u_{xx})_x = 0, \qquad u_t + (\tfrac{1}{2}u^2 + h - \nu u_x)_x = 0. \tag{34}$$

Of course, these are the KBB system (2) itself. However, while the first equation in (33) is the representation of conservation of mass, the second equation is not that for momentum conservation, as this should have the term $(hu)_t$ and not u_t. Instead, the second equation is in effect the conservation of the Bernoulli expression.

Averaging Eqs. (34) over the periodic solution $u(\theta)$, $h(\theta)$ of the unperturbed KB system we obtain two modulation equations

$$\overline{h}_t + (\overline{hu})_x = 0, \qquad \overline{u}_t + (\tfrac{1}{2}\overline{u^2} + \overline{h})_x = 0. \tag{35}$$

We note that the dissipative term drops out of the averaged conservation laws (35) so the dissipation can only enter other modulation equations. But if the full perturbed modulation system admits the travelling solutions of the form $f(x-ct)$, c being constant, then the equations (35) must also admit such a solution. Substitution of $\overline{h} = \overline{h}(x-ct)$, $\overline{u} = \overline{u}(x-ct)$ into Eqs. (35) yields

$$-c\overline{h} + \overline{hu} = -A, \qquad -c\overline{u} + \tfrac{1}{2}\overline{u^2} + \overline{h} = B, \tag{36}$$

A, B being constants. Let the established undular bore satisfy the boundary conditions (3) at infinity. Then considering Eqs. (36) at $x - ct \to \pm\infty$ we obtain

$$h_2 u_2 - h_1 u_1 = c(h_2 - h_1)\,, \qquad \tfrac{1}{2}(u_2^2 - u_1^2) + h_2 - h_1 = c(u_2 - u_1) \tag{37}$$

which can be conveniently represented as

$$c = u_1 + h_2\sqrt{\frac{2}{h_1 + h_2}}, \qquad u_2 = u_1 + (h_2 - h_1)\sqrt{\frac{2}{h_1 + h_2}}. \tag{38}$$

Thus, we have obtained an important restriction on the admissible family of the initial steps that may be eventually resolved into a single frictional undular bore with no additional (rarefaction) wave involved (cf. analogous condition (32) for dissipationless case). These conditions agree with the formal jump conditions obtained from the same two conservation laws for h and u of the ideal shallow water dynamics (6).

However, it is well known that the usual "physical" jump conditions providing the mass and the *momentum* balance across the bore have the form (Benjamin, Lighthill 1954, Whitham 1974)

$$c = u_1 + h_2\sqrt{\frac{h_1 + h_2}{2h_1 h_2}}, \quad u_2 = u_1 + (h_2 - h_1)\sqrt{\frac{h_1 + h_2}{2h_1 h_2}}. \tag{39}$$

The discrepancy between the jump conditions (38) and (39) occurs due to absence of the proper momentum conservation law for the KB-Boussinesq system. This apparent disagreement, however, can be resolved by noticing that considered within the range of physical applicability of the KBB system, i.e. for small $h_2 - h_1 \ll h_1$ the transition conditions (38) and (39) are asymptotically equivalent. In both cases we have

$$c \approx u_1 + \sqrt{h_1} + \frac{3}{4}\frac{h_2 - h_1}{\sqrt{h_1}}, \qquad u_2 \approx u_1 + \frac{h_2 - h_1}{\sqrt{h_1}}. \tag{40}$$

In the next section we will show that the transition conditions in the form (38) also follow from the exact (non-periodic) travelling wave solution of the full KBB system (2).

3 Travelling wave solution of the KBB system: steady undular bore

Here we shall study a steady travelling wave solution of the KBB system (2), i.e. we introduce the ansatz

$$u = u(\theta), \qquad h = h(\theta), \qquad \theta = x - ct. \tag{41}$$

Its substitution into (2) leads to equations, which can be readily integrated once to give

$$\begin{aligned} -ch + hu + \tfrac{1}{4}u_{\theta\theta} &= -A, \\ -cu + \tfrac{1}{2}u^2 + h &= \nu u_\theta + B, \end{aligned} \tag{42}$$

where A and B are again integration constants. Then the boundary conditions (3) yield the relations (38) and also

$$\begin{aligned} A &= \frac{h_1 h_2 (u_2 - u_1)}{h_2 - h_1}, \\ B &= u_1(\tfrac{1}{2}u_1 - c) + h_1 = u_2(\tfrac{1}{2}u_2 - c) + h_2 \end{aligned} \tag{43}$$

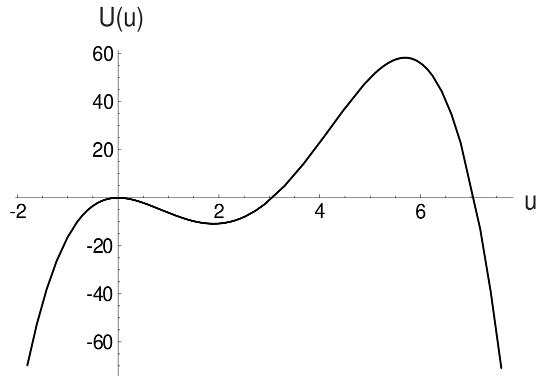

Figure 6: Potential (45) for an effective particle motion according to the Newton equation (44).

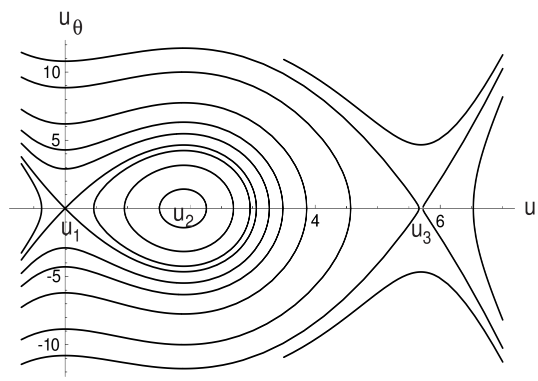

Figure 7: Phase plane for an effective particle motion in the potential (45).

Upon eliminating h from Eqs. (42) one arrives at the equation

$$u_{\theta\theta} + 4\nu(u-c)u_\theta = 4u(\tfrac{1}{2}u - c)(u-c) - 4B(u-c) - 4A\,. \tag{44}$$

This can be considered as the Newton equation for a motion of a particle with a coordinate u (with θ playing the role of time) in the potential

$$U(u) = -\tfrac{1}{2}u^4 + 2cu^3 - 2(c^2 - B)u^2 - 4(Bc - A)u + \text{constant} \tag{45}$$

whose plot is shown in Fig. 6. A phase plane $(u,\, u_\theta)$ for the undamped oscillator corresponding to the potential (45) is shown in Fig. 7. There are three critical points $(u_1, 0)$, $(u_2, 0)$, $(u_3, 0)$: the point $(u_2, 0)$ is stable and the points $(u_1, 0)$ and $(u_3, 0)$ are unstable. The closed trajectories around the centre $(u_2, 0)$ correspond to a periodic motion and the separatrix corresponds to a soliton. Introducing small damping (the second term in the left-hand side of the Eq. (44)) leads to an aperiodic oscillatory solution with the phase trajectory starting from the saddle point $(u_1, 0)$ and eventually arriving after spiralling at the stable focus at $(u_2, 0)$. This trajectory corresponds to a steady undular bore.

The spatial oscillatory structure implied by this phase trajectory is the following. The large amplitude oscillations starting at $(u_1, 0)$ correspond to the soliton train at the leading

edge of the undular bore and the small amplitude oscillations in the vicinity of $(u_2, 0)$ correspond to its trailing edge. It should also be noted that the configuration of the potential in Fig. 7 corresponds to the undular bore moving to the right. A bi-directional KBB system allows also an alternative configuration of the potential curve (45) with the double roots at $(u_3, 0)$. This would reverse the picture so that the phase trajectory would start at the saddle point $(u_3, 0)$ and after spiralling would again arrive at the potential minimum at $(u_2, 0)$. This trajectory corresponds to the left-propagating undular bore

The oscillatory profile of the bore can be found by numerical integration of Eq. (44). However, a quite effective analytical theory can be developed on the basis of the Whitham modulation theory. The idea of the Whitham description of the frictional undular bore is to replace a weakly aperiodic motion of the particle in a given fixed potential with the asymptotically equivalent conservative motion in the potential which is slowly deformed. The Whitham equations then describe equivalent slow deformations of the potential. An advantage of the Whitham description in the case of perturbed integrable dynamics is that it utilizes the underlying integrable structure, and allows us to obtain the modulation equations using a universal technique based on powerful methods from finite-gap integration theory. At the same time, a straightforward application of the perturbation procedure would require very specific and lengthy calculations.

It should also be noted that the modulations in the undular bore are not solely due to the dissipation. Rather, weak dissipation modifies the structure of the dissipationless undular bore.

4 Modulation equations

The derivation of the Whitham modulation equations for the KBB system (2) is based on the complete integrability of the unperturbed KB system

$$\begin{aligned} &h_t + (hu)_x + \tfrac{1}{4}u_{xxx} = 0\,, \\ &u_t + uu_x + h_x = 0\,. \end{aligned} \tag{46}$$

That is, on the possibility to represent it as a compatibility condition of two linear equations for an auxiliary function ψ:

$$\psi_{xx} = \mathcal{A}\psi, \qquad \psi_t = -\frac{1}{2}\mathcal{B}_x\psi + \mathcal{B}\psi_x \tag{47}$$

with

$$\mathcal{A} = \left(\lambda - \frac{1}{2}u\right)^2 - h, \qquad \mathcal{B} = -\left(\lambda + \frac{1}{2}u\right), \tag{48}$$

where λ is a spectral parameter. In the framework of this approach, the parameters λ_i entering the periodic solution (15), (16) of Eqs. (46) have the following meaning. The second order differential equation (47) has two basis solutions ψ^+ and ψ^- from which we can build the so-called 'squared basis function'

$$g = \psi^+\psi^-. \tag{49}$$

It is easy to show that it satisfies the equation

$$g_{xxx} - 2\mathcal{A}_x g - 4\mathcal{A}g_x = 0, \tag{50}$$

which after multiplication by $g/2$ can be integrated once to give

$$\frac{1}{2}gg_{xx} - \frac{1}{4}g_x^2 - \mathcal{A}g^2 = -P(\lambda), \tag{51}$$

where the integration constant denoted by $-P(\lambda)$ can depend on the spectral parameter λ. The second equation (47) gives

$$g_t = \mathcal{B}g_x - \mathcal{B}_x g. \tag{52}$$

In the finite-gap integration method (see, e.g. Kamchatnov, 2000), the periodic solutions are distinguished by the condition that $P(\lambda)$ be a polynomial in λ. Then g as a function of λ should also be a polynomial in λ. The one-phase periodic solution (15), (16) corresponds to

$$P(\lambda) = \prod_{i=1}^{4}(\lambda - \lambda_i) = \lambda^4 - s_1\lambda^3 + s_2\lambda^2 - s_3\lambda + s_4 \tag{53}$$

and

$$g = \lambda - \mu. \tag{54}$$

Then from Eqs. (51) and (52) we find at once the relations (15) as well as the equation for μ,

$$\mu_\theta = 2\sqrt{P(\mu)} \tag{55}$$

whose integration yields (16).

As we see, the parameters λ_i are the zeroes of the polynomial $P(\lambda)$ which determine the periodic solution in the finite-gap integration method. At the same time, the parameters λ_i are the most convenient modulation variables in terms of which the Whitham modulation equations assume the diagonal Riemann form (21) for the unperturbed KB equations (46) or its counterpart (33) for perturbed, KBB dynamics (2). As was shown by Kamchatnov (2004), if the evolution equations are written symbolically as

$$u_{m,t} = K_m(u_n, u_{n,x}, \dots,) + R_m(x, t, u_n, u_{n,x}, \dots,), \qquad m, n = 1, \dots, N, \tag{56}$$

where the functions K_m correspond to the "leading", integrable part of the system, and the perturbation terms R_m can be slow functions of x and t and can also depend on the field variables u_n and their space derivatives, then the perturbed Whitham equations have the form

$$\frac{\partial\lambda_i}{\partial t} + v_i\frac{\partial\lambda_i}{\partial x} = \frac{1}{\langle 1/g\rangle \prod_{j\neq i}(\lambda_i - \lambda_j)} \sum_{m=1}^{N}\sum_{l=0}^{A_m}\Big\langle \frac{\partial\mathcal{A}}{\partial u_m^{(l)}}\frac{\partial^l R_m}{\partial x^l} g\Big\rangle_i, \quad i = 1, \dots, M, \tag{57}$$

where

$$v_i = -\frac{\langle \mathcal{B}/g\rangle_i}{\langle 1/g\rangle_i}, \quad i = 1, \dots, M\,. \tag{58}$$

Here the angle brackets denote averaging over the proper interval of x, M is the degree of the polynomial $P(\lambda)$, A_m is the order of the highest derivative $u_m^{(A_m)}$ in $\mathcal{A}$, and the index for the bracket means that λ is put equal to λ_i.

In our case for the KBB system (2) we have $M = 4$ and

$$\begin{aligned} &N = 2: \quad u_1 = h, \quad u_2 = u;\\ &R_1 = 0, \quad R_2 = \nu u_{xx}, \quad A_2 = 0, \quad \partial\mathcal{A}/\partial u = u/2 - \lambda. \end{aligned} \tag{59}$$

Hence the perturbation terms on the right-hand side of Eqs. (57) take the form

$$\rho_i = \frac{\nu\langle(\lambda_i - u/2)u_{xx}g\rangle_i}{\langle 1/g\rangle_i \prod_{j\neq i}(\lambda_i - \lambda_j)}, \quad i = 1, 2, 3, 4. \tag{60}$$

For the one-phase modulated solutions of our present interest, one can replace averaging in (57), (58), (60) with the averaging over the unperturbed periodic family (55). Then, using (54), (54) we have

$$\left\langle \frac{1}{g} \right\rangle_i = \frac{1}{2L}\oint \frac{d\mu}{(\lambda_i - \mu)\sqrt{P(\mu)}} = -\frac{2}{L}\frac{\partial L}{\partial \lambda_i}, \tag{61}$$

where the wavelength L is given by (22). Further, taking account of the relations $u = s_1 - 2\mu$, $\mu_x = 2\sqrt{P(\mu)}$, $u_{xx} = -4dP/d\mu$ we have

$$\left\langle \frac{\mathcal{B}}{g} \right\rangle_i = \frac{1}{2L}\oint \frac{\mu - \lambda_i - s_1/2}{(\lambda_i - \mu)\sqrt{P(\mu)}}d\mu = \frac{s_1}{L}\frac{\partial L}{\partial \lambda_i} - 1\,, \tag{62}$$

$$\begin{aligned} \langle(\lambda_i - u/2)gu_{xx}\rangle_i &= \frac{1}{L}\oint \left(\lambda_i - \frac{u}{2}\right) g u_{xx}\frac{dx}{d\mu}d\mu \\ &= -\frac{2}{L}\oint \frac{(\lambda_i - s_1/2 + \mu)(\lambda_i - \mu)}{\sqrt{P(\mu)}}\frac{dP}{d\mu}d\mu \\ &= -\frac{4}{L}\oint \left(-\mu^2 + \tfrac{1}{2}s_1\mu - \tfrac{1}{2}s_1\lambda_i + \lambda_i^2\right)\frac{d\sqrt{P(\mu)}}{d\mu}d\mu \\ &= -\frac{8}{L}\oint (\mu - s_1/4)\sqrt{P(\mu)}\,d\mu. \end{aligned} \tag{63}$$

Then, the characteristic velocities (58) take the form

$$v_i = \frac{s_1}{2} - \frac{L}{2}\left(\frac{\partial L}{\partial \lambda_i}\right)^{-1} \tag{64}$$

coinciding with the unperturbed case (21), while the perturbation terms in the Whitham equations (33) are given by

$$\rho_i = \frac{8\nu}{(\partial L/\partial \lambda_i)\prod_{j\neq i}(\lambda_i - \lambda_j)}\int_{\lambda_2}^{\lambda_3}(\mu - s_1/4)\sqrt{P(\mu)}\,d\mu. \tag{65}$$

The integral here can be evaluated in terms of complete elliptic integrals. However, the resulting expression is very complicated and it is easier to deal with its unevaluated form (65). The Whitham equations (33), (65) determine the evolution of the parameters λ_i due to nonuniform modulation of the wave, and the small of effect viscosity. It is natural to expect that for the boundary conditions (3) the modulated wave will asymptotically, as $t \to \infty$, tend to the steady undular bore solution described in Section 3. In the next section we shall find the corresponding stationary solution of the Whitham equations.

5 Steady solution of the Whitham equations

We look for the solution of the Whitham equations (33) in the form

$$\lambda_i = \lambda_i(\theta), \qquad \theta = x - ct, \tag{66}$$

so that we must have

$$-c\frac{d\lambda_i}{d\theta} + \left(\frac{s_1}{2} - \frac{L}{2\partial L/\partial\lambda_i}\right)\frac{d\lambda_i}{d\theta} = \frac{8\nu}{(\partial L/\partial\lambda_i)\prod_{j\neq i}(\lambda_i - \lambda_j)}\int_{\lambda_2}^{\lambda_3}(\mu - s_1/4)\sqrt{P(\mu)}\,d\mu. \tag{67}$$

Motivated by the meaning of $s_1/2$ as the phase velocity one can suggest that equations (67) can be split in the following way:

$$c = \frac{s_1}{2} = \text{const} \tag{68}$$

and

$$\frac{d\lambda_i}{d\theta} = \frac{Q}{\prod_{j\neq i}(\lambda_i - \lambda_j)}, \tag{69}$$

where the factor

$$Q = -\frac{8\nu}{L}\int_{\lambda_2}^{\lambda_3}(\mu - s_1/4)\sqrt{P(\mu)}\,d\mu \tag{70}$$

is the same for all $i = 1, 2, 3, 4$.

For Eqs. (68), (69) to be consistent with Eq. (67) s_1 must be an integral of equations (69). In fact, we will show that the special structure of these equations provides actually three integrals s_1, s_2, s_3. This statement can be proved with the use of the Jacobi identities (Jacobi 1884), which follow at once in the most convenient for us from the obvious identity

$$\sum_{i=1}^{n}\frac{\prod_{j\neq i}(\lambda - \lambda_j)}{\prod_{j\neq i}(\lambda_i - \lambda_j)} = 1 \tag{71}$$

where in the left-hand side we have a polynomial in λ of the degree $n-1$ which is equal to unity at n points $\lambda = \lambda_i$, $i = 1, \ldots, n$, and hence is equal to unity identically. Then equating the coefficients of λ^m at both sides of (71) we get $n-1$ identities for $m \neq 0$,

$$\sum_{i=1}^{n}\frac{1}{\prod_{j\neq i}(\lambda_i - \lambda_j)} = 0, \quad \sum_{i=1}^{n}\frac{\sum'_j\lambda_j}{\prod_{j\neq i}(\lambda_i - \lambda_j)} = 0, \quad \sum_{i=1}^{n}\frac{\sum'_{j,k}\lambda_j\lambda_k}{\prod_{j\neq i}(\lambda_i - \lambda_j)} = 0, \ldots \tag{72}$$

where prime means that all terms with the factor λ_i are omitted in the corresponding sum, and the last identity for $m = 0$ can be written in the form

$$\sum_{i=1}^{n}\frac{1}{\lambda_i\prod_{j\neq i}(\lambda_i - \lambda_j)} = \frac{(-1)^{n-1}}{s_n}, \tag{73}$$

where $s_n = \prod_i\lambda_i$. In our case $n = 4$ and Eqs. (69) and (72) yield

$$\begin{aligned}
&\frac{ds_1}{d\theta} = Q\sum_{i=1}^{4}\frac{1}{\prod_{j\neq i}(\lambda_i - \lambda_j)} = 0, \quad \frac{ds_2}{d\theta} = Q\sum_{i=1}^{4}\frac{\sum'_j\lambda_j}{\prod_{j\neq i}(\lambda_i - \lambda_j)} = 0, \\
&\frac{ds_3}{d\theta} = Q\sum_{i=1}^{4}\frac{\sum'_{j,k}\lambda_j\lambda_k}{\prod_{j\neq i}(\lambda_i - \lambda_j)} = 0,
\end{aligned} \tag{74}$$

that is the system (69) has $n-1=3$ integrals of motion

$$s_1 = \mathrm{const}, \quad s_2 = \mathrm{const}, \quad s_3 = \mathrm{const}. \tag{75}$$

Thus, in the steady solution only the last coefficient s_4 varies with θ according to the equation

$$\frac{ds_4}{d\theta} = \sum_{i=1}^{4} \frac{s_4}{\lambda_i}\frac{d\lambda_i}{d\theta} = s_4 Q \sum_{i=1}^{4} \frac{1}{\lambda_i \prod_{j\neq i}(\lambda_i - \lambda_j)} = -Q, \tag{76}$$

where we have used the identity (73).

Now, the zeroes λ_i, $i = 1,2,3,4$, are the solutions of the algebraic equation

$$P(\lambda) = \prod_{i=1}^{4}(\lambda - \lambda_i) = \lambda^4 - s_1\lambda^3 + s_2\lambda^2 - s_3\lambda + s_4 = 0 \tag{77}$$

ordered according to

$$\lambda_1 \le \lambda_2 \le \lambda_3 \le \lambda_4 \tag{78}$$

and for given s_1, s_2, s_3 they can be considered as known functions of s_4. As a result, we arrive at the single first order differential equation

$$\frac{ds_4}{d\theta} = \frac{8\nu}{L}\int_{\lambda_2(s_4)}^{\lambda_3(s_4)} (\mu - s_1/4)\sqrt{P(\mu)}\,d\mu, \tag{79}$$

where L is given by Eq. (22). For the solution under study, $\lambda_i = \lambda_i(s_4)$, hence in the right-hand side of (79) we have a known function of s_4. The constants s_1, s_2, s_3 can be expressed in terms of the initial parameters h_1, h_2, u_1, as in (38). To this end, we compare the equation

$$u_\theta^2 = u^4 - 4cu^3 + 4(c^2 - B)u^2 + 8(Bc - A)u + \mathrm{const}, \tag{80}$$

following from Eq. (44) with $\nu = 0$, with the equation (see (55))

$$\mu_\theta^2 = 4(\mu^4 - s_1\mu^3 + s_2\mu^2 - s_3\mu + s_4)\,. \tag{81}$$

Then taking account of the relation $u = s_1 - 2\mu$, these must coincide with each other, and so we find that

$$s_1 = 2c, \quad s_2 = c^2 - B, \quad s_3 = -(A + Bc), \tag{82}$$

where c is given by (38) and (see (43))

$$A = h_1 h_2\sqrt{\frac{2}{h_1 + h_2}}, \quad B = h_1 - \tfrac{1}{2}u_1^2 - u_1 h_1\sqrt{\frac{2}{h_1 + h_2}}. \tag{83}$$

To determine the interval within which the variable s_4 can vary, we notice that at the leading and trailing edges of the undular bore the polynomial $P(\lambda)$ has double roots, that is, its discriminant D vanishes. Hence, the limiting values of s_4 must be the roots of the equation (see, e.g. Fricke 1924)

$$D = g_2^3 - 27g_3^2 = 0, \tag{84}$$

where

$$
\begin{aligned}
g_2 &= s_4 - \tfrac{1}{4}s_1 s_3 + \tfrac{1}{12}s_2^2, \\
g_3 &= \tfrac{1}{6}s_2 s_4 + \tfrac{1}{48}s_1 s_2 s_3 - \tfrac{1}{216}s_2^3 - \tfrac{1}{16}s_3^2 - \tfrac{1}{16}s_1^2 s_4
\end{aligned}
\tag{85}
$$

are invariants of the polynomial $P(\lambda)$. Equation (84) is cubic with respect to s_4 and has three roots $s_4^{(1)} < s_4^{(2)} < s_4^{(3)}$. The variable μ (and, hence, u) oscillates with finite amplitude as long as $P(\lambda)$ has three real roots. Hence, s_4 can vary between the two smaller zeroes of the discriminant D,

$$
s_4^{(1)} < s_4 < s_4^{(2)}. \tag{86}
$$

Thus, all parameters in Eq. (79) are completely determined and can be expressed in terms of h_1, h_2, u_1, so that dependence of s_4 on θ can be found by integration of Eq. (79) in the interval Eq. (86) with the initial condition

$$
\frac{ds_4}{d\theta} = s_4^{(2)} \quad \text{at} \quad \theta = \theta_0, \tag{87}
$$

where we assume that the leading edge of the bore is located at $\theta = \theta_0$.

At the trailing edge Eq. (79) reduces approximately to

$$
\frac{ds_4}{d\theta} \cong \text{const} \cdot (\lambda_3(s_4) - \lambda_2(s_4))^2. \tag{88}
$$

Since in the vicinity of $s_4^{(1)}$ we have

$$
\lambda_2(s_4),\ \lambda_3(s_4) \propto \sqrt{s_4 - s_4^{(1)}}, \tag{89}
$$

then here

$$
\frac{ds_4}{d\theta} \cong C \cdot (s_4 - s_4^{(1)}) \tag{90}
$$

which gives at once

$$
s_4 - s_4^{(1)} \propto \exp(C\theta) \tag{91}
$$

where C is some constant proportional to ν. Thus, we see that the trailing edge is formally located at $\theta \to -\infty$, but with exponential accuracy we can take the width of the bore as

$$
\Delta \cong \frac{\text{const}}{\nu}. \tag{92}
$$

The asymptotic analogous to Eq. (91) has been obtained in (Gurevich and Pitaevskii 1987) and (Myint and Grimshaw 1995) for the KdV-Burgers equation.

At the leading edge we have a soliton solution (15), (19) with $\lambda_i = \lambda_i(s_4^{(2)})$. Its centre corresponds to $\mu(0) = \lambda_2$ and hence Eqs. (15) give the values of the velocity u_s and elevation h_s at the centre of the leading soliton:

$$
u_s = s_1 - 2\lambda_2(s_4^{(2)}), \quad h_s = \tfrac{1}{4}s_1^2 - s_2 - 2(\lambda_2(s_4^{(2)}))^2 + s_1\lambda_2(s_4^{(2)}). \tag{93}
$$

The dependence of all values entering the right-hand parts of Eqs. (93) on the initial parameters is given by Eqs. (82) – (85).

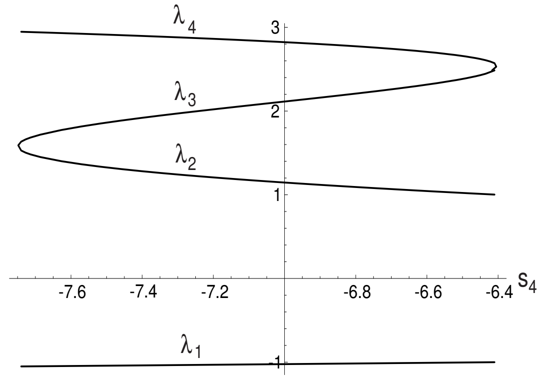

Figure 8: The Riemann invariants as functions of s_4.

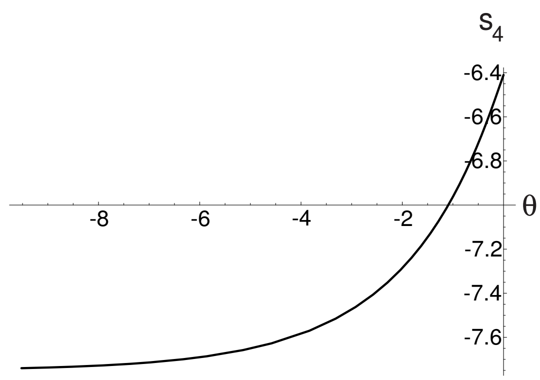

Figure 9: Dependence of s_4 on θ in the bore.

To illustrate the developed theory, let us make some calculations and draw corresponding plots for specific parameters of the undular bore. Although the asymptotic approximation used in the derivation of the KBB system is consistent with the shallow water dynamics only for small values of the initial step $h_2 - h_1 \ll h_1$ (see Eqs. (39),(40)) it is instructive to consider the problem with noticeably distinct initial parameters h_1 and h_2 for a better exposure of the details of the oscillatory structure. We choose $\nu = 0.1$ and

$$u_1 = 0, \quad h_1 = 1, \quad h_2 = 4. \tag{94}$$

Then we get

$$u_2 = 1.90, \quad c = 2.53, \quad A = 2.53, \quad B = 1.0 \tag{95}$$

and

$$s_1 = 5.06, \quad s_2 = 5.4, \quad s_3 = -5.06. \tag{96}$$

Equation (84) gives the limits for s_4,

$$s_4^{(1)} = -7.74, \qquad s_4^{(2)} = -6.4. \tag{97}$$

Solving Eq. (77) for λ_i, we find the Riemann variables λ_i as functions of s_4 and the corresponding plot is shown in Fig. 8. Integration of (79) leads to dependence of s_4 on θ shown in

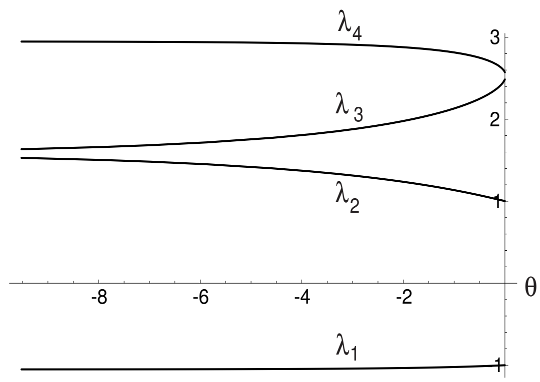

Figure 10: The Riemann invariants as functions of θ.

Fig. 9. Substitution of this dependence into $\lambda_i(s_4)$, $i = 1, 2, 3, 4$, yields the Riemann variables as functions of θ depicted in Fig. 10. As we see, the trailing edge is located at $\theta \to -\infty$ where $\lambda_2(\theta)$ and $\lambda_3(\theta)$ tend to the same limit $\lambda_2(s_4^{(1)}) = \lambda_3(s_4^{(1)})$. Finally, substitution of the slowly varying Riemann variables into (15) yields the profiles of velocity $u(\theta)$ and water elevation $h(\theta)$ in the bore; see Figs. 11 and 12, respectively. Obviously, the "camel hump" form of the lead soliton of the elevation profile in Fig. 12 is due to the properties of the Kaup-Boussinesq system rather than actual properties of shallow water solitary waves. One should note, though, that within the range of applicability of the Boussinesq approximation (i.e. for small enough initial steps) this deviation of the soliton shape from the regular shallow water soliton profile ceases to be visible.

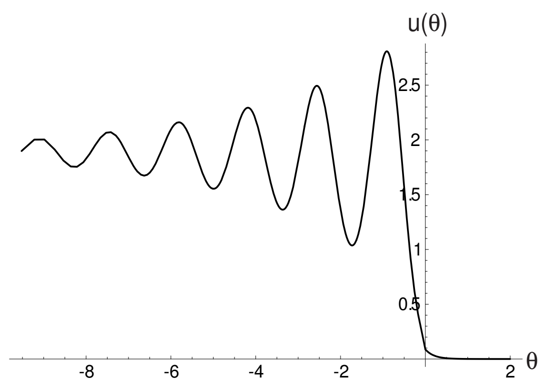

Figure 11: Velocity profile in the steady undular bore.

One can see, that despite the different quantitative description, frictional and conservative shallow water undular bores are structurally similar in many respects (cf. El, Grimshaw & Pavlov 2001). There are, however, substantial qualitative differences. In particular: (i) the conservative undular bore expands in time while the frictional undular bore with the Burgers-like dissipative term asymptotically reaches a steady profile propagating with a single velocity c; (ii) the transition relations across the frictional and the dissipationless undular bores are different: for the frictional undular bore the transition relation (38) coincides with the jump condition following from the conservation laws (34), while in the dissipationless case (see (32)) it coincides with the simple wave relation for the ideal shallow-water equations.

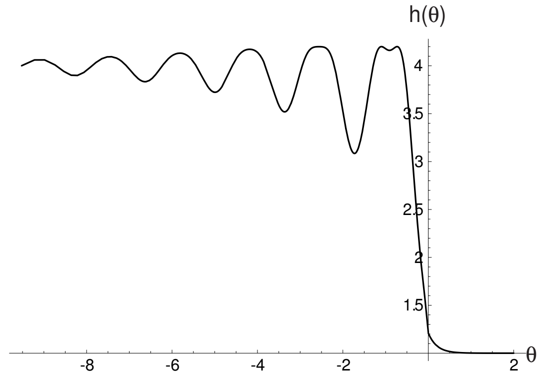

Figure 12: Elevation profile in the steady undular bore.

We note in conclusion, that it is clear that the method of solution of the perturbed integrable Whitham equations used in this paper is essentially based on the special structure of the perturbation term (65) where the integral is actually a function of just a single parameter s_4. Similar structure, however, can appear due to other than Burgers-type perturbation of the original integrable equation. Generally, this is the case for any form of perturbation term leading, after averaging, to the integrand in (65) containing only symmetric functions of the Riemann variables λ_i rather than individual λ_i's. Then one can find sufficient number of integrals of the stationary Whitham equations to reduce the system to a single equation.

6 Conclusions

The formation of a shallow-water frictional undular bore has been studied analytically using the Kaup-Boussinesq system modified by a small friction term. The main stages of the undular bore formation from the step-like initial profile were considered and the analytic solutions were constructed for the initial unsteady (dissipationless) and final steady (frictional) stages of the undular bore development, using the Whitham method. The perturbed integrable Whitham equations for the Kaup-Boussinesq-Burgers system were derived using the methods of finite-gap integration. It was shown that the stationary solution of the Whitham equations describing modulations in the steady undular bore is consistent with the jump conditions following from the original conservation laws for the KBB system.

The theory developed in this paper shows that the Whitham method provides a general effective approach to describe frictional undular bores in perturbed integrable systems, and can be used in different physical contexts provided the dissipation is small enough to not prevent the generation of nonlinear dispersive waves, but sufficient to balance the combined action of nonlinearity and dispersion at large times.

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